Mathematics II

(English course)

Second semester, 2012/2013

Exercises (1)

1. Let

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 0 & -1 \\ 1 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 3 & 0 & -2 \\ 0 & -7 & -1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 & -2 \\ -3 & -5 \end{bmatrix}, \qquad E = \begin{bmatrix} 4 & -2 & -3 & 2 \end{bmatrix}.$$

For which pairs among the above matrices is matrix multiplication possible? Compute the product for every admissible pair.

2. For each of the systems of linear equations below, find all its solutions or show that the system does not admit any solution.

(a) $$\begin{cases} x - 2y = 0 \\ -3x - z = 2 \\ x + 5y + z = 1 \end{cases}$$

(b) $$\begin{cases} x - 2y + z = -1 \\ -3x - z = 2 \end{cases}$$

(c) $$\begin{cases} x + 2y + z = -1 \\ y + z = 2 \\ x + y + z = 1 \\ 2x + 2y + z = -2 \end{cases}$$

(d) $$\begin{cases} -x + 3y + z = 3 \\ x - y + z = 1 \\ x - 2y + z = 1 \\ 2x + 2y + z = -2 \end{cases}$$

(e) $$\begin{cases} x - y + 3z + w = 1 \\ 2x + y - z + 2w = -1 \\ x + 2y - 4z + w = 3 \end{cases}$$

3. Consider the matrices

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -3 & 0 & -1 \\ 1 & 5 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 3 & 0 & -2 \\ 0 & -7 & -1 \end{bmatrix}, \qquad C = \begin{bmatrix} 1 & -2 \\ -3 & -5 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 1 \\ 3 & -1 & 0 \end{bmatrix}, \qquad E = \begin{bmatrix} 0 & -2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -2 \\ 0 & 0 & -3 & 0 \end{bmatrix}.$$

(a) For each of the matrices above, find its inverse or show that it does not admit inverse.

(b) Which of the matrices above have determinant? Find the determinants and the determinant of the inverse matrix whenever possible.

4. For each of the following matrices, find the eigenvalues. For every real eigenvalue, find the associated eigenvectors.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 2 \\ 4 & 1 \end{bmatrix}, \qquad C = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix},$$

$$D = \begin{bmatrix} -2 & -1 \\ 5 & 2 \end{bmatrix}, \qquad E = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 0 & -2 \\ -1 & 2 & 3 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$G = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \qquad H = \begin{bmatrix} 3 & -2 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \qquad I = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & -17 & 8 \end{bmatrix},$$

5. Prove the following statements:

(a) $\alpha \in \mathbb{C}$ is an eigenvalue of A if and only if it is an eigenvalue of A^T.

(b) $\alpha \in \mathbb{C}$ is an eigenvalue of A if and only if $-\alpha$ is an eigenvalue of $-A$.

(c) If α is an eigenvalue of A and $\mathrm{Det}(A) \neq 0$, then $\alpha \neq 0$ and $\frac{1}{\alpha}$ is an eigenvalue of A^{-1}.

(d) If α is an eigenvalue of A, then α^k is an eigenvalue of A^k.

6. Let A be a real 3×3 matrix such that $A^3 = Id$.

(a) What are the eigenvalues of A?

(b) Give an example of such a matrix other than $A = Id$.

7. Let A be a real 3×3 matrix with two-dimensional kernel.

   For each of the following assertions, find a proof if true or a counterexample if false.

   (a) $\lambda = 0$ is an eigenvalue of A with algebraic multiplicity equal to 2.

   (b) $\mathrm{Tr}(A)$ is equal to an eigenvalue of A.

8. Show that the matrix

$$A = \begin{bmatrix} 0 & 5 & 1 & 0 \\ 5 & 0 & 5 & 0 \\ 1 & 5 & 0 & 5 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

   has two positive and two negative eigenvalues (counting algebraic multiplicities).

   Remark: You don't need to compute the eigenvalues.